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A Non-associative Deformation of Yang-Mills Gauge Theory

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Abstract

An ansatz is presented for a possible non-associative deformation of the standard Yang-Mills type gauge theories. An explicit algebraic structure for the deformed gauge symmetry is put forward and the resulting gauge theory developed. The non-associative deformation is constructed in such a way that an apparently associative Lie algebraic structure is retained modulo a closure problem for the generators. It is this failure to close which leads to new physics in the model as manifest in the gauge field kinetic term in the resulting Lagrangian. A possible connection between this model and quantum group gauge theories is also investigated.

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I. INTRODUCTION

Recently there has been considerable interest in the construction of *quantum group gauge theories* (QGGTs) [1–11], where a quantum group plays the role of the gauge group. One source of motivation for this work has been the suggestion that relaxing the rigid structure of the Lie group based gauge theories may lead to new explanations for fundamental theoretical problems such as spontaneous symmetry breaking and quark confinement.

Lie group theory, as the predominant mathematical tool for the analysis of physical symmetries, has been extraordinarily successful in providing a unified description of many aspects of particle physics. In particular the currently accepted descriptions of the strong, weak, and electromagnetic interactions, have the common theme of an underlying gauge symmetry described by a Lie group. Despite this success there continue to be difficulties in unifying the various forces, and indeed in explaining some of the features of the individual forces such as those mentioned above. It would seem reasonable to consider the possibility that a full unification of the fundamental forces may require a mathematical structure beyond groups. In the first instance we might intuitively expect that such a structure would be generalisation of the Lie group. The quantum group approach, regardless of how it was originally conceived, is clearly a construction of this type wherein the Lie group gauge symmetry is replaced by a more general quantum group symmetry which reduces to the standard case in the limit of some parameter.

The transition from theories based on Lie group internal symmetry spaces to those where the symmetry is that of a quantum group has, however, proved to be rather problematic. The main efforts have focussed on keeping the classical form of the gauge transformations. The gauge potential A transforms as follows,

$$A \longrightarrow A' = UAU^{-1} - \frac{i}{g}(\partial U)U^{-1}, \tag{1}$$

where U is chosen to be an element of a quantum group. The difficulty as described by Aref'eva and Arutyunov [9] is to determine the relevant differential calculus and also the

algebra from which the gauge potentials A should be drawn to ensure that A' also belongs to that algebra. Recently it has been claimed [9] that it is only possible to present an algebraic group gauge potential based on $U_q(N)$, and that groups such as $SU_q(N)$ are not allowed. It has also been claimed [6] that if the gauge fields have values in $U_q(g)$, the quantum universal enveloping algebra of the Lie algebra g, then the resulting gauge theory will be isomorphic to the non-deformed theory if the base space is classical spacetime. The implication being that to obtain non-trivial results an underlying quantum space must be considered [7]. Although the situation is far from clear at this stage it would appear that the most general QGGTs require detailed analysis of both the differential calculus on quantum groups (see for example Woronowicz [12,13]), and also the non-commutative geometric structure of quantum spaces.

With this complex situation in mind, we present in this letter an alternative approach to the generalisation of the standard Yang-Mills type gauge theories. Our approach will be to extend the standard (Lie) gauge group while retaining as much of the Lie algebraic structure as possible. Consequently this will allow construction of the gauge theory to proceed in the standard manner, with the resulting theory being a deformation of the standard one. Gauge theories based on extensions of simple Lie groups such as non-semisimple Lie groups have been considered recently [14]. In this letter we will take a larger step to a theory where the underlying gauge "group" is non-associative. A non-associative algebra has no group structure in the normal sense but by considering the algebra as a deformation of a Lie algebra we can obtain the form of the resulting deformation of the gauge field. For this reason the theory will apparently break the gauge symmetry, but only when this symmetry is assumed to be of the Lie group form. It is in this sense that we can regard the resulting theory as one involving a higher "non-associative gauge symmetry".

Our justification for considering non-associativity as the mechanism for extending the Lie group structure is twofold. Firstly, non-associative algebras have been linked with a number of interesting gauge groups. The exceptional GUT groups, such as E_6, and the internal symmetry group of the anomaly free heterotic string $E_8 \times E_8$ have in common the fact that they are automorphism groups of the non-associative exceptional Jordan algebra M_3^8

of 3×3 matrices over the octonions [15–17]. Günaydin & Gürsey [28,29] also used the fact that $SU(3)$ is a subgroup of the automorphism group of the octonions to obtain a theory of quark confinement, which was subsequently extended by Dixon [30,31]. Although the gauge groups in these cases are not strictly non-associative this common link is suggestive of a deeper underlying non-associative "symmetry".

The non-associative octonions, the last in the sequence of four division algebras of the Hurwitz theorem, have also been linked to spacetime symmetries in 10 dimensions. The Lorentz "group" in this case is essentially $SL(2)$ over the octonions [24,20]. Consequently octonionic spinors [24,27,26,25] have also been linked to 10 dimensional spacetime and the Green-Schwarz superstring finds a natural formulation in terms of the exceptional Jordan algebra M_3^8 [19–23]. These correlations, and the association of supersymmetry in ten dimensions with the octonions [24,32,25], suggests that a non-associative internal symmetry may be particularly relevant for theories in ten spacetime dimensions. Finally, on a more technical point, non-associative structures such as 3-cocycles have been linked with chiral anomalies in field theories [34,33,35] and therefore removal of such problems may also require a non-associative description.

Our second justification for considering a non-associative deformation, and as a motivation behind the algebraic structure we shall choose, is that it provides a framework for considering tensor product gauge groups, i.e. $G = A \otimes B \otimes C \otimes \ldots$, where there is some *coupling* between the algebras of the different elements. The coupling then implies that the gauge group is no longer a direct product and therefore a more complete unification of the groups $A, B, C, \ldots$ into the group G is achieved.

The major problem with considering a non-associative gauge theory is that a gauge group in the normal sense does not exist, due to the non-associativity. Our technique for dealing with this owes its inspiration partly to the gauge theories considered by Waldron & Joshi [36], and Lassig & Joshi [37], where the gauge algebra was that of the octonions. The octonions form a non-associative alternative algebra, and thus a generalisation of the Lie group approach to gauging is required. Our approach will not specifically involve octonions

but for clarity it is worth reviewing the form of the octonionic gauge theory making reference to how it relates to the generalisation of gauge theories.

It is well known that the octonionic units can be represented in terms of (associative) left and right matrices in the bimodular representation. The details of how this representation is obtained have been considered previously (see [17,36,37]) and will not be reproduced here. In this representation the non-associativity is manifest in the inability of either the left matrix or right matrix algebra to *close*. When considered in isolation the left matrices can be considered as generators of a Lie algebra where the extra generators required to close the algebra are missing. In the octonion case these missing generators are replaced by a coupling between the left and right matrices. i.e. the left matrix algebra is "closed" via a coupling to the right matrices, and vice versa.

If we were to consider only the left matrices as a gauge algebra [36] then the "missing" generators in the Lie algebra, and their construction via coupling to another algebra (the right matrices), give the new physics which will become apparent in the resulting gauge field Lagrangian. This can be made more explicit by noting that the left matrices in the bimodular octonion representation are also generators of the $SO(8)$ symmetry group. Thus the octonionic symmetry can be visualised as some particular observable channels of the $SO(8)$ symmetry. Importantly calculations can still be made as though the full $SO(8)$, i.e. Lie group, symmetry were present. The restriction on the generators available will then lead to the new physics in the system.

The octonionic case discussed qualitatively above was used to indicate how the methodology of using a non-associative generalisation of the Lie algebra structure allows the Lie structure of the gauge group to be retained modulo the closure problem. The fact that the standard Lie algebraic calculational techniques can still be used (cf. QGGTs) implies that we can consider a standard Yang-Mills type theory. Having considered this possibility qualitatively we propose a possible algebraic structure for the gauge symmetry in Section 2, and develop the corresponding gauge theory in Section 3. A possible correspondence between this approach and QGGTs is also considered in this section.

II. A PARAMETERISED NON-ASSOCIATIVE ALGEBRA

Standard Yang-Mills type gauge theories based on internal symmetries described by non-Abelian Lie groups have been extraordinarily successful in particle physics with electroweak theory and QCD being the most prominent examples. Some possible reasons for generalising this structure were mentioned in Section 1, however it is clear that we would wish to retain most of the nice features of these theories. Thus we would expect the generalised theory to reduce to the standard theory in the limit of some parameter, as in QGGTs. We can achieve this, and our previously mentioned aim to consider coupling in tensor product gauge groups, in our non-associative formalism with the following algebraic structure of the gauge group.

Consider M sets of N generators which we can represent in the following matrix.

$$(T)_{ji} = \begin{pmatrix} T_{1i} \\ T_{2i} \\ \vdots \\ T_{Mi} \end{pmatrix} = \begin{pmatrix} T_{11} & & \cdots & T_{1N} \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \vdots \\ T_{M1} & & \cdots & T_{MN} \end{pmatrix}. \tag{2}$$

For generality we allow the possibility that the algebras are not all of equal dimension. Then for a particular set of generators it may be that

$$T_{(p)(1)} \ldots T_{(p)(r)} \neq 0 \qquad T_{(p)(r+1)} \ldots T_{(p)(N)} = 0. \tag{3}$$

We regard each set of generators T_{pi}, for fixed $p \in 1..M$, as the generators of a simple Lie algebra which may or may not *close*. We represent this in the following way:

$$[T_i^p, T_j^p] = f_{ijk}^p T_k^p + \sum_{n=1}^{M} \sigma_n^p [T_j^n, T_i^p], \tag{4}$$

where $p \in 1..M$ and the σ_n^p are constants. In this representation the parameters σ_n^p, for $n \in 1..M$, determine the closure of each set of generators T_{pi}. For a given $p \in 1..M$, if $\sigma_n^p = 0 \;\; \forall n$ then the generators T_{pi} close and we have a normal associative Lie algebra. If, however, there exist $n \in 1..M, \;\; n \neq p$ such that $\sigma_n^p \neq 0$ then the generators T_{pi} do not close and this is manifest in some *mixing* between the sets of generators. This *nonlinearity* will imply non-associativity for the algebra of the set T_{pi}.

This can be made explicit by considering the *Jacobi function*

$$J(a,b,c) = [a,[b,c]] + [b,[c,a]] + [c,[a,b]], \tag{5}$$

where $J(a,b,c) = 0$ for a,b,c elements of an associative algebra. For the case at hand, for fixed p,

$$J(T_i^p, T_j^p, T_k^p) = \sigma_n^p \left([T_i^p, [T_k^n, T_j^p]] + [T_j^p, [T_i^n, T_k^p]] + [T_k^p, [T_j^n, T_i^p]] \right). \tag{6}$$

Thus associativity is restored for $\sigma_n^p = 0 \quad \forall n \in 1..M$ or alternatively if all the sets of generators commute. We can write this explicitly in terms of the *associator*,

$$(a,b,c) = (ab)c - a(bc), \tag{7}$$

by noting

$$\epsilon^{ijk}(a_i, a_j, a_k) = J(a_i, a_j, a_k). \tag{8}$$

Thus we have an algebraic structure where the non-associativity is manifest in the inability of the set of Lie algebraic generators T_{pi} to close. This ensures that the generators will still have a matrix representation, and we can retain the nice features of Lie algebras, and to some extent its group structure. The non-associativity can be "turned on" by closing the algebra via mixing with generators from other sets.

III. NON-ASSOCIATIVE GAUGING

We consider a Yang-Mills type theory where the gauge "group" has the algebraic structure $\mathcal{A}$ considered in the previous section:

$$[T_i^p, T_j^p] = f_{ijk}^p T_k^p + \sigma_n^p [T_j^n, T_i^p], \tag{9}$$

where the σ_n^p are constants, and the summation over $n = 1..M$ is implicit. We assume $p \in [1..M]$ and that the indices $i,j,k \in [1..N]$ label the elements of each particular set of generators. The constants σ_n^p parameterise the level of non-associativity.

The Lie algebraic structure retained in this algebra allows the Yang-Mills gauge theory to be developed in the standard way. We introduce the following matter fields:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_D \end{pmatrix}, \tag{10}$$

where D is the dimension of the representation of $\mathcal{A}$. Since the constituent generators of $\mathcal{A}$ are equivalent to those of a Lie algebra we can consider exponentiating to produce a structure which, in the limit $\sigma_n^p \rightarrow 0 \quad \forall n$, will correspond to a Lie group. This allows consideration of the following standard gauge transformation,

$$\psi(x) \longrightarrow \psi'(x) = e^{-i\tau_i^p \theta_p^i(x)}\psi(x) = U(\theta_p)\psi(x), \tag{11}$$

where $p \in 1..M$ is not summed. The particular representation of $\mathcal{A}$, in this case given by $D \times D$ matrices τ_i^p, satisfies the algebra,

$$[\tau_i^p, \tau_j^p] = f_{ijk}^{p'}\tau_k^p + \sigma_n^{p'}[\tau_j^n, \tau_i^p], \tag{12}$$

and we impose

$$\mathrm{Tr}(\tau_i^p \tau_j^p) = \frac{1}{2}\delta_{ij} \qquad \forall p \in 1..M, \tag{13}$$

on the adjoint representation via the relevant normalisation. Since we will henceforth work only with this representation the primes in $f_{ijk}^{p'}$ and $\sigma_n^{p'}$ will be suppressed.

We can now proceed in the standard way by defining the covariant derivative as

$$\mathcal{D}_\mu^p = \partial_\mu + igA_\mu^p(x), \tag{14}$$

where

$$A_\mu^p(x) = {}^i A_\mu^p(x)\tau_i^p, \tag{15}$$

and summation over $i = 1..N$, but not $p \in 1..M$, is implicit. This explicitly introduces the vector gauge fields ${}^iA^p_\mu(x)$ which, in the standard case, would ensure that the Lagrangian density $\mathcal{L}$ is invariant under local gauge transformations. The $A^p_\mu(x)$ transform as

$$A^p_\mu \longrightarrow A^{p'}_\mu = UA^p_\mu U^{-1} - \frac{i}{g} U\partial_\mu U^{-1}. \tag{16}$$

On evaluation for infinitesimal transformations,

$$A^{p'}_\mu = \left({}^iA^{p'}_\mu\right)\tau^p_i + i\sigma^p_n \theta^i_p(x)\, {}^jA^p_\mu[\tau^n_j, \tau^p_i], \tag{17}$$

where

$${}^iA^{p'}_\mu = {}^iA^p_\mu + f^{p'}_{ijk}\theta^i_p\, {}^kA^p_\mu + \frac{1}{g}\partial_\mu \theta^i_p(x), \tag{18}$$

is the algebraically closed part in the usual form. Thus the non-associativity is manifest in the inability of the transformation to close. We note that for $\sigma^p_n = 0 \ \forall n$ the transformation does close as required.

The antisymmetric curvature tensor can be defined in the normal way:

$$\begin{aligned} F^p_{\mu\nu} &= -\frac{i}{g}[\mathcal{D}^p_\mu, \mathcal{D}^p_\nu] \\ &= \partial_{[\mu}A^p_{\nu]} + ig[A^p_\mu, A^p_\nu] \\ &= {}^iF^p_{\mu\nu}\tau^p_i + i\sigma^p_n g\, {}^jA^p_\mu\, {}^kA^p_\nu[\tau^n_k, \tau^p_j], \end{aligned} \tag{19}$$

where again,

$${}^iF^p_{\mu\nu} = \partial^i_{[\mu}A^p_{\nu]} - gf^p_{ijk}\, {}^jA^p_\mu\, {}^kA^p_\nu, \tag{20}$$

is the algebraically closed part.

We can now evaluate the gauge field kinetic term in the Lagrangian density. We obtain:

$$\begin{aligned} \mathcal{L}^p_{gauge} &= -\frac{1}{2}\mathrm{Tr}(F^p_{\mu\nu}F^{p\mu\nu}) \\ &= -\frac{1}{2}\, {}^iF^p_{\mu\nu}\, {}^iF^{p\mu\nu} - \frac{1}{2}i\sigma^p_n g\, {}^jA^p_\mu\, {}^kA^p_\nu\, {}^tF^p_{\mu\nu}(\mathrm{Tr}([\tau^n_k, \tau^p_j]\tau^p_t)) \\ &\quad -\frac{1}{2}i\sigma^p_n g\, {}^uA^p_\mu\, {}^vA^p_\nu\, {}^iF^p_{\mu\nu}(\mathrm{Tr}(\tau^p_i[\tau^n_v, \tau^p_u])) \\ &\quad +\frac{1}{2}(\sigma^p_n)^2 g^2\, {}^jA^p_\mu\, {}^kA^p_\nu\, {}^uA^p_\mu\, {}^vA^p_\nu(\mathrm{Tr}([\tau^n_k, \tau^p_j][\tau^n_v, \tau^p_u])). \end{aligned} \tag{21}$$

This Lagrangian represents the kinetic term for a gauge field where the algebra of the primary generators (the p'th set) is altered by mixing with external generators. We note that the terms corresponding to this mixing are suppressed by factors of σ_n^p, and thus by setting these coupling constants to zero the non-associativity is turned off and a standard Yang-Mills gauge kinetic term results.

We note that this Lagrangian is however biased in favour of the p'th set of generators, with the other sets entering via the nonlinear algebraic relations. This is the relevant situation if we are considering a small coupling between a primary algebra and a secondary algebra, however the implicit bias towards the p^{th} set used so far may be artificial in others. We can obtain a symmetric Lagrangian density for the gauge field by summing the contributions $\forall p \in 1..M$. This implies the gauge transformation is now

$$\psi(x) \to \psi'(x) = e^{-i\tau_i^p \theta_p^i(x)} \psi(x) \tag{22}$$

where now both i and p are summed. The relations for the covariant derivative, Eq. 14, and the gauge fields, Eq. 15, can now be reinterpreted with p summed over $1..M$. The covariant derivative now takes on a form similar in appearance to that encountered with tensor product gauge groups. The difference being in that here there exists the possibility for coupling between the components. The curvature tensor in symmetrised form is then,

$$F_{\mu\nu} = \partial_\mu A_\nu^p - \partial_\nu A_\mu^p + ig[A_\mu^p, A_\nu^p] + ig[A_\mu^r, A_\nu^s], \tag{23}$$

where $p, r, s = 1..M, \quad r \neq s$. Thus we have

$$F_{\mu\nu} = {}^iF_{\mu\nu}^p \tau_i^p + ig\sigma_n^p \, {}^jA_\mu^p \, {}^kA_\nu^p [\tau_k^n, \tau_j^p] + ig \, {}^tA_\mu^r \, {}^uA_\nu^s [\tau_t^r, \tau_u^s], \tag{24}$$

where ${}^iF_{\mu\nu}^p$ is given by Eq. 20, i, j, k, t, u are summed over $1..N$, and p is now summed over $1..M$. This symmetric formulation fundamentally alters the covariant derivative and thus the limit $\sigma_n^p \to 0$ alone no longer reduces the theory to one with no coupling. If, however, all the generators in different sets commute then the Lagrangian for each set will decouple and have the standard form.

This will obviously lead to a more complicated gauge kinetic term for the symmetrised Lagrangian which we include for completeness,

$$
\begin{aligned}
\mathcal{L}_{gauge} &= -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \\
&= -\frac{1}{2}\ {}^{a}F^{p}_{\mu\nu}\ {}^{f}F^{q}_{\mu\nu}\mathrm{Tr}[\tau^{p}_{a}\tau^{q}_{f}] - \frac{1}{2}i\sigma^{q}_{m}g\ {}^{g}A^{q}_{\mu}\ {}^{h}A^{q}_{\nu}\ {}^{a}F^{p}_{\mu\nu}\mathrm{Tr}[\tau^{p}_{a}[\tau^{m}_{h},\tau^{q}_{g}]] \\
&\quad -\frac{1}{2}ig\ {}^{i}A^{t}_{\mu}\ {}^{j}A^{u}_{\nu}\ {}^{a}F^{p}_{\mu\nu}\mathrm{Tr}[\tau^{p}_{a}[\tau^{t}_{i},\tau^{u}_{j}]] - i\sigma^{p}_{n}g\ {}^{b}A^{p}_{\mu}\ {}^{c}A^{p}_{\nu}\ {}^{f}F^{q}_{\mu\nu}\mathrm{Tr}[[\tau^{n}_{c},\tau^{p}_{b}]\tau^{q}_{f}] \\
&\quad +\frac{1}{2}\sigma^{p}_{n}\sigma^{q}_{m}g^{2}\ {}^{b}A^{p}_{\mu}\ {}^{c}A^{p}_{\nu}\ {}^{g}A^{q}_{\mu}\ {}^{h}A^{q}_{\nu}\ \mathrm{Tr}[[\tau^{n}_{c},\tau^{p}_{b}][\tau^{m}_{h},\tau^{q}_{g}]] \\
&\quad +\frac{1}{2}\sigma^{p}_{n}g^{2}\ {}^{b}A^{p}_{\mu}\ {}^{c}A^{p}_{\nu}\ {}^{i}A^{t}_{\mu}\ {}^{j}A^{u}_{\nu}\ \mathrm{Tr}[[\tau^{n}_{c},\tau^{p}_{b}][\tau^{t}_{i},\tau^{u}_{j}]] \\
&\quad -\frac{1}{2}ig\ {}^{d}A^{r}_{\mu}\ {}^{e}A^{s}_{\nu}\ {}^{f}F^{q}_{\mu\nu}\ \mathrm{Tr}[[\tau^{r}_{d},\tau^{s}_{e}]\tau^{q}_{f}] \\
&\quad +\frac{1}{2}\sigma^{q}_{m}g^{2}\ {}^{d}A^{r}_{\mu}\ {}^{e}A^{s}_{\nu}\ {}^{g}A^{q}_{\mu}\ {}^{h}A^{q}_{\nu}\ \mathrm{Tr}[[\tau^{r}_{d},\tau^{s}_{e}][\tau^{m}_{h},\tau^{q}_{g}]] \\
&\quad +\frac{1}{2}g^{2}\ {}^{d}A^{r}_{\mu}\ {}^{e}A^{s}_{\nu}\ {}^{i}A^{t}_{\mu}\ {}^{j}A^{u}_{\nu}\ \mathrm{Tr}[[\tau^{r}_{d},\tau^{s}_{e}][\tau^{t}_{i},\tau^{u}_{j}]]
\end{aligned} \tag{25}
$$

where $a,b,c,d,e,f,g,h,i,j = 1..N$ and $p,q,r,s,t,u = 1..M,\ r \neq s,\ t \neq u$. This gives us the full gauge field kinetic term in the general case, which represents the result of the non-associative deformation of the gauge group. With regard to renormalisation, superficially this Lagrangian density has no terms of higher than quartic power in the gauge fields. However a full consideration of renormalisability would require calculations to loop level which have not been considered as this is very much a toy model at this stage.

The analysis has been of a general nature thus far. We will now indicate how the general algebraic structure introduced in Section 2 includes various subalgebras which have been previously been considered as possible gauge algebras.

A. Specific Cases

1. Associative Lie Algebras

The standard form gauge groups are realised trivially when

$$\sigma^{p}_{n} = 0 \quad \forall n \in 1..M. \tag{26}$$

The generator sets then decouple and there is no need for symmetrisation. The normal Yang-Mills type Lagrangian density results. i.e.

$$\mathcal{L}_{gauge} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu}. \tag{27}$$

In practice since there are so many nice features of these theories, we might expect that the situation of most interest in the low energy regime might be when the σ^p_n are small, and thus the theory approaches the Lie algebra case. The Lie algebra structure of the gauge group would then only be slightly perturbed by the induced non-associativity.

2. Octonionic Algebras

Gauge theories based on an octonionic gauge algebra have been considered before [36,37] and they represent the first non-trivial instance of the general algebraic structure considered. This is observed by using the left/right matrix bi-representation for octonions. This implies two sets of generators, the left and right matrices in this case. The gauge theories mentioned above, in the notation of Lassig & Joshi [37], can be realised when

$$\sigma^p_n = \begin{cases} 2 \ \text{ for } \ n, p = 1, 2; \quad n \neq p \\ 0 \ \text{ otherwise} \end{cases}, \tag{28}$$

and

$$f^p_{ijk} = (-1)^{p-1} \frac{i}{2} \epsilon_{ijk}, \tag{29}$$

where ϵ_{ijk} is the anti-symmetric tensor for octonions where, using the standard multiplication table, $\epsilon_{ijk} = 1$ for each cycle $ijk = 123, 145, 176, 246, 257, 347, 365$. Thus there are two coupled sets of generators, which are the λ_i, ρ_i matrices of [36] and [37], representing the left and right matrices of the bimodular representation.

3. Quantum Groups

As mentioned in the introduction quantum group gauge theories have been the subject of considerable recent interest. In these theories the process of gauging was altered to

generate a more general quantum group symmetry. In contrast, in this discussion we have retained the standard Yang-Mills machinery allowing the standard symmetry to be broken by the deformation. Despite this difference in approach we show in this section how the general algebraic structure can be linked to the quantum universal enveloping algebra of a Lie algebra.

For concreteness we consider the quantum universal enveloping algebra $U_q(su(2))$ in the Drinfel'd-Jimbo basis (see for example [38,39]), whose generators satisfy the following relations

$$[J_\pm, J_3] = \mp J_\pm \qquad [J_+, J_-] = [2J_3]_q, \tag{30}$$

where, in the notation of Macfarlane [39], the q-integers are given by

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{31}$$

Writing $J_\pm = J_1 \pm iJ_2$ we have

$$\begin{aligned} [J_1, J_2] &= \frac{1}{2}i[2J_3]_q \\ [J_2, J_3] &= \frac{1}{2}i(2J_1) \\ [J_3, J_1] &= \frac{1}{2}i(2J_2). \end{aligned} \tag{32}$$

To make this tractable, we assume $q = 1+\delta$, where $|\delta| \ll 1$, but $\delta \neq 0$. Then we can expand the q-integers as a power series and we have

$$\begin{aligned} q^{2J_3} - q^{-2J_3} &= (1+\delta)^{2J_3} - (1+\delta)^{-2J_3} \\ &\approx \sum_{n=1}^{\infty} C_n J_3^n, \end{aligned} \tag{33}$$

where

$$C_n = C_n(\delta^n, \delta^{n+1}, \delta^{n+2}, \ldots) \tag{34}$$

is a constant, which will be convergent for small δ. Note that in this case the above approximation becomes exact. As an aside we also note that for $U_q(su(2))$ C_n is only non-zero for odd n.

In this case we can now rewrite Eq.s 32 as

$$
\begin{aligned}
[J_1, J_2] &= i\left(\tfrac{C_1}{2(q-q^{-1})}\right) J_3 + \tfrac{i}{2(q-q^{-1})} \textstyle\sum_{n=2}^{\infty} C_n J_3^n \\
[J_2, J_3] &= iJ_1 \\
[J_3, J_1] &= iJ_2.
\end{aligned} \tag{35}
$$

Thus the algebra can be represented in the following form

$$[J_i, J_j] = f_{ijk} J_k + N_{ijk}, \tag{36}$$

with the nonzero antisymmetric structure constants

$$f_{ijk} = i\epsilon_{ijk} + i\epsilon_{ijk}\delta_{k3}\left(\frac{C_1}{2(q-q^{-1})} - 1\right), \tag{37}$$

where ϵ_{ijk} is the Levi-Civita antisymmetric tensor, and the extra term N_{ijk}, which represents the inability of the algebra to close, is given by

$$N_{ijk} = i\epsilon_{ijk}\delta_{k3}\frac{1}{2(q-q^{-1})}\sum_{n=2}^{\infty} C_n J_3^n. \tag{38}$$

We now assume that there exist generators K_{ni}, where $n \in 2..\infty$ and $i, j \in 1, 2, 3$, such that

$$[K_{nj}, J_i] = \epsilon_{ijk}\delta_{k3} J_k^n. \tag{39}$$

Thus we have

$$N_{ijk} = \frac{i}{2(q-q^{-1})}\sum_{n=2}^{\infty} C_n [K_{nj}, J_i]. \tag{40}$$

Therefore we can finally represent the algebra as

$$[J_i, J_j] = f_{ijk} J_k + \frac{i}{2(q-q^{-1})} C_n [K_{nj}, J_i], \tag{41}$$

where the summation over $n = 2..\infty$ is implicit. This is exactly the form required for our general procedure, and thus the q-algebra is realised in this form when we interpret

$$
\begin{aligned}
M &= \infty \\
\sigma_n &= \frac{i}{2(q-q^{-1})} C_n.
\end{aligned} \tag{42}
$$

It is therefore apparent that to fully represent the quantum group as a gauge group in this manner requires mixing between an infinite number of generators! We should also note that the algebra is not symmetric and the required gauge term in the Lagrangian would also be unsymmetric with respect to all the generators. This is due to the bias towards the set $\{J_i\}$ as a result of our construction.

Clearly it would appear that this formalism is not particularly useful for considering quantum gauge groups, however for practical calculational purposes we can note that if $q \sim 1$ then the terms in the infinite series for C_n would quickly tend to zero. Then a truncation of the series would seem reasonable and the constants σ_n could be explicitly evaluated giving, in effect, a perturbation to a given order in $(1-q)$. The perturbation series would then interpolate to some extent between the Lie algebra and the full quantum group, for the case when $q \sim 1$.

IV. CONCLUDING REMARKS

In this letter we have considered a possible formalism for the analysis of a gauge theory based on a "group" with a non-associative algebraic structure. The gauge algebra presented was obtained initially as a generalisation of the standard Lie algebraic structure, where coupling between different Lie algebras is allowed. This non-associative formalism is apparently quite different from other Lie algebra generalisations such as the infinite dimensional Kac-Moody algebras.

Gaining inspiration from the bimodular representation of octonions the non-associativity inherent in the algebra, for ease of calculation, is implicit as a closure problem for the algebra of the generators. The analysis of the resulting gauge theory has been somewhat cursory, and in particular renormalisability for this toy model has not been considered.

Finally, we would like to point out that the viewpoint taken in this letter has resulted in a theory where the gauge symmetry is broken via the non-associative coupling between the sets of generators. This formalism was used to explicitly show the new physics obtained

by deforming the symmetry, and this explicit gauge symmetry breaking is manifest when the constants σ_n^p become non-zero and the individual generator sets no longer close. An alternative viewpoint is that although a Lie group symmetry has been broken a higher "non-associative" symmetry is retained. This viewpoint requires a significant alteration to the current understanding of what constitutes a gauge symmetry. If this seems to radical then an alternative to our construction could be considered where a full (generalised) symmetry is retained. This is the approach taken with QGGTs and would require deformation of the standard gauge transformations, and most likely the form of the gauge field Lagrangian density. The Lie group type symmetry is then deformed and reduces to the standard one in the limit of some parameter. It is clear that obtaining such a theory where a full symmetry, in something approaching the Lie group sense, is retained would be worthwhile, and this is under investigation.

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